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Equivalence of Boolean Algebras and Pre A*-Algebras

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ABSTRACT:*In this paper we present definition of Boolean algebra and Boolean sub algebra, examples, theorems on Boolean algebras. Definition of pre A*algebras, examples and we show that Boolean algebras generates Pre-A*algebras, correspondence between Boolean algebras and Pre-A*algebras.*

KEY WORDS: *Boolean algebra, Pre -A*algebra.*

I. INTRODUCTION:

E. G. MANES introduced an Ada based on C-Algebras introduced by Fernando Guzman and Craig C. Squir . P. Koteswara Rao introduced the concept of A^*-Algebras analogous to the E. G. Manes ,Adas. J. Venkateswara Rao introduced the concept of Pre A*-Algebra analogous to the C-Algebra , as a reduct of A*-Algebra. He studied Pre A*-Algebras and their sub directly irreducible representations. It was established that 2 = {0; 1}and 3 = {0; 1; 2 }are the only sub directly irreducible Pre A*-Algebras and that every Pre A*-Algebra can be embedded in 3^x, for some set x. Also proved that a Pre A*-Algebra can be made into an A*-Algebra by imposing one binary operation and one unary operation and he obtained a sufficient condition for a Pre A*-Algebra to become an A*-Algebra. In this paper we studied Boolean Algebras generates Pre A*-Algebras and One –one correspondence between Boolean algebras and Pre A*-Algebras

II. BOOLEAN ALGEBRA:

2.1.Definition: An algebra (B, ∧, ∨, $(-)^{\sim}$, 0, 1) is called a Boolean algebra if it satisfies: for every a, b, $c \in$ B

i) $a \wedge a = a,\ a \vee a = a$

ii) $a \wedge b = b \wedge a,\ a \vee b = b \vee a$

iii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

$a \vee (b \vee c) = (a \vee b) \vee c$

iv) $(a \wedge b) \vee a = a$

$(a \vee b) \wedge a = a$

v) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

vi) $a \vee 0 = a$, $a \wedge 1 = a$

vii) $a \wedge a^1 = 0$, $a \vee a^1 = 1$

2.1.Example: 2 = {0, 1} with ∧, ∨, $(-)^{\sim}$ defined by

∧	0	1
0	**0**	**1**
1	**1**	**1**

∨	0	1
0	**0**	**0**
1	**0**	**1**

X	X^1
0	**1**
1	**0**

2.2.Example: Suppose X is a set, P(X) is a Boolean Algebra under the operation of set intersection ∩, union ∪ and complementation as $(-)^{\sim}$, ϕ, X as ∧, ∨, $(-)^{\sim}$, 0 and 1.

2.3. Example: The class of all logical statements form Boolean algebra under the operations and (∧), or (∨), not (~), fallacy f and tautology t as ∧, ∨, $(-)^{\sim}$, 0 and 1.

2.4.Theorem : Suppose (B, ∧, ∨, $(-)^1$, 0, 1) is a Boolean Algebra, then

i) $a \vee b = 1,\ a \wedge b = 0 \Rightarrow b = a^1$
ii) $a^{\|} = a$
iii) $(a \vee b)^{|} = a^{|} \wedge b^{|}, (a \wedge b)^{|} = a^{|} \vee b^{|}$
iv) $a \wedge b^{|} = 0 \Leftrightarrow a \wedge b = a$
v) $0^{|} = 1, 1^{|} = 0$
vi) $a \wedge (a^{|} \vee b) = a \wedge b$

Proof:

i) $b = b \vee 0 = b \vee (a \wedge a^{|})$
$= (b \vee a) \wedge (b \vee a^{|})$
$= 1 \wedge (b \vee a^{|})$
$= (b \vee a^{|})$
$a^{|} = a^{|} \vee 0 = a^{|} \vee (a \wedge b)$
$= (a^{|} \vee a) \wedge (a^{|} \vee b)$
$= 1 \wedge (a^{|} \vee b)$
$= (a^{|} \vee b) = b \wedge a^{|} = b$
$\therefore a^{|} = b$

ii) $a^{\|} \wedge a^{|} = 0, a^{|} \vee a^{\|} = 0$
$a \wedge a^{|} = 0, a \vee a^{|} = 0$
$\Rightarrow$ $a, a^{\|}$ are inverses of $a^{|}$
$\therefore a = a^{\|}$

iii) $(a \vee b) \wedge (a^{|} \wedge b^{|}) = 0$
$(a \vee b) \vee (a^{|} \wedge b^{|}) = 1$
$\therefore (a^{|} \vee b^{|})^{|} = a^{|} \wedge b^{|}$
$\therefore (a \wedge b)^{|} = a^{|} \vee b^{|}$

iv) Suppose $a \wedge b^{|} = 0$
$a = a \wedge 1 = a \wedge (b \vee b^{|})$

$= (a \wedge b) \vee (a \wedge b^{|})$
$= (a \wedge b) \vee 0$
$= (a \wedge b)$
$\therefore a \wedge b$
Suppose $a \wedge b = a$
$a \wedge b^{|} = 0 \vee (a \wedge b^{|})$
$= (a \wedge a^{|}) \vee (a \wedge b^{|})$
$= a \wedge (a^{|} \vee b^{|})$
$= a \wedge (a \wedge b)^{|}$
$= a \wedge a^{|} = 0$

$\therefore a \wedge b^{|} = 0$

$\because 0 \vee 1 = 1, 0 \wedge 1 = 0$

$\Rightarrow 1^{|} = 0$ and $0^{|} = 1$

v) $a \wedge (a^{|} \vee b) = (a \wedge a^{|}) \vee (a \wedge b)$
$= 0 \vee (a \wedge b)$
$= a \wedge b$
$\therefore a \wedge (a^{|} \vee b) = a \wedge b$
Similarly $a \vee (a^{|} \wedge b) = a \vee b$

III. Pre A*- Algebras:

3.1 Definition: An algebra $(A, \wedge, \vee, (-)^{\sim})$ satisfying

(a) $x^{\sim\sim} = x$,for all $x \in A$,

(b) $x \wedge x = x$, for all $x \in A$,

(c) $x \wedge y = y \wedge x$, for all $x, y \in A$,

(d) $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim}$, for all $x, y \in A$,

(e) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, for all $x,y,z \in A$

(f) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,for all $x,y,z \in A$,
(g) $x \wedge y = x \wedge (x^{\sim} \wedge y)$, for all $x,y,z \in A$,
is called a Pre A*-algebra.

3.2 Example: 3 = {0, 1, 2} with operations $A, \wedge, \vee, (-)^{\sim}$ defined below is a Pre A*-algebra

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

x	$x^{\sim}$
0	1
1	0
2	2

3.3 Note : $(2, \wedge, \vee, (-)^{\sim})$ is a Boolean algebra. So every Boolean algebra is a Pre A*-Algebra

3.4 Lemma: Let A be a pre A*-Algebra and $a \in A$ be an identity for $\wedge$, then $a^{\sim}$ is an identity for $\vee$, a unique if it exists, and is denoted by 1 and $a^{\sim}$ by 0.i.e
(i) $a \wedge x = x$ for all $x \in A$ (ii) $a^{\sim} \vee x = x$ for all $x \in A$

3.5 Lemma: Let A be a pre A*- Algebra with 1 and 0 and let $x, y \in A$

(i)If $x \vee y = 0$, implies $x = 0$. (ii) If $x \vee y = 1$, implies $x \vee x^{\sim} = 1$.

3.6. Theorem: Let $(B, \wedge, \vee, (-)^{\sim}, 0, 1)$ be a Boolean algebra, then
$A(B) = \{(a,b)/a,b \in B, a \wedge b = 0\}$ becomes a Pre-A*algebra, where $\wedge, \vee, (-)^{\sim}$ are defined as follows.
For $a = (a_1, a_2)$, $b = (b_1, b_2) \in A(B)$
i) $a \wedge b = (a_1b_1, a_1b_2 + a_2b_1 + a_2b_2)$
ii) $a \vee b = (a_1b_1 + a_1b_2 + a_2b_1, a_2b_2)$
iii) $a^{\sim} = (a_2, a_1)$
iv) $I = (1,0)$, $0 = (0,1)$, $2 = (0,0)$

Proof: Clearly $(a^{\sim} \wedge b^{\sim})^{\sim} = a \vee b$
(i) $(a^{\sim})^{\sim} = (a_2, a_1)^{\sim}$
$= (a_1, a_2) = a$
$\therefore (a^{\sim})^{\sim} = a$
(ii) $a \wedge a = a$
Now $a \wedge a = (a_1, a_2) \wedge (a_1, a_2)$
$= (a_1a_1, a_1a_2 + a_2a_1 + a_2a_2)$
$= (a_1, 2a1a2 + a_2)$
$= (a_1, a_2) = a$
$\therefore a \wedge a = a$

(iii) $a \wedge b = b \wedge a$, for all $a, b \in A(B)$

$a \wedge b = (a_1, a_2) \wedge (b_1, b_2)$

$= (a_1b_1, a_1b_2 + a_2b_1 + a_2b_2)$

$b \wedge a = (b_1, b_2) \wedge (a_1, a_2)$

$= (b_1a_1, b_2a_1 + b_1a_2 + b_2a_2)$

$= (a_1b_1, a_1b_2 + a_2b_1 + a_2b_2)$

$\therefore a \wedge b = b \wedge a$

(iv) $(a \wedge b)^{\sim} = a^{\sim} \vee b^{\sim}$

Now $a \wedge b = (a_1, a_2) \wedge (b_1, b_2)$

$= (a_1b_1, a_1b_2 + a_2b_1 + a_2b_2)$

$(a \wedge b)^{\sim} = (a_1b_2, a_2b_1 + a_2b_2 + a_1b_1)$

$a^{\sim} \vee b^{\sim} = (a_1, a_2)^{\sim} \vee (b_1, b_2)^{\sim}$

$= (a_2, a_1) \vee (b_2, b_1)$

$= (a_2b_2 + a_2b_1 + a_1b_2, a_1b_1)$

$= (a_1b_2 + a_2b_1 + a_2b_2, a_1b_1)$

$\therefore (a \wedge b)^{\sim} = a^{\sim} \vee b^{\sim}$

(v) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ is clear

(vi) $a \wedge (b \wedge c) = (a \wedge b) \wedge (a \wedge c)$ is clear

(vii) $a \wedge b = a \wedge (a^{\sim} \wedge b)\ \forall a, b, c \in B(A)$

$(a \wedge b) = (a_1, a_2) \wedge (b_1, b_2)$

$= (a_1b_1, a_1b_2 + a_2b_1 + a_2b_2)$

$a \wedge b = (a_1, a_2) \wedge (b_1, b_2)$

$= (a_1b_1, a_1b_2 + a_2b_1 + a_2b_2)$

$(a^{\sim} \vee b) = (a_2, a_1) \vee (b_2, b_1)$

$= (a_2b_1 + a_2b_2 + a_1b_1, a_1b_2)$

$a \wedge (a^{\sim} \vee b) = (a_1, a_2) \wedge (a_2b_1 + a_2b_2 + a_1b_1, a_1b_2)$

$\therefore$ A(B) is a Pre-A*algebra.

3.7 Theorem: If A is Pre-A*algebra and B is a Boolean algebra then

(i) $B(A(B)) \cong B$ (ii) $A(B(A)) \cong A$

Proof: (i) is clear

To prove (ii) define

$f: A \to A(B(A))$ by $f(a) = (a_\pi, a_\pi^{\sim})$

$a = b \Leftrightarrow a_\pi = b_\pi;\ a_\pi^{\sim} = b_\pi^{\sim}$

$\Leftrightarrow (a_\pi, a_\pi^{\sim}) = (b_\pi, b_\pi^{\sim})$

$\Leftrightarrow f(a) = f(b)$

$\therefore f$ is well defined.

Let $\alpha \in A(B(A))$

$\Rightarrow \alpha = (a_\pi, a^{\#})$, where $a_\pi, a^{\#} \in B(A)$

$a_\pi \wedge a^{\#} = 0$

Let $a(x) = a_\pi * a^{\#}$

$f(a) = ((a_\pi * a_\pi^{\#}), (a_\pi * a_\pi^{\#})^{\sim})$

$= (a_\pi, (a_\pi)^{\sim} \wedge a^{\#})$

$= (a_\pi, a^{\#}) = \alpha$

$\therefore f(a) = \alpha$

$\therefore f$ is onto

$f(a \wedge b) = ((a \wedge b_\pi), a \wedge b_\pi^{\sim})$

$= (a_\pi b_\pi, a_\pi^{\sim} b_\pi + a_\pi b_\pi^{\sim} + a_\pi^{\sim} b_\pi^{\sim})$

$= (a_\pi, a_\pi^{\sim}) \wedge (b_\pi, b_\pi^{\sim})$

$= f(a) \wedge f(b)$

$\therefore f(a \wedge b) = f(a) \vee f(b)$

$f(a \vee b) = ((a \vee b)_{\pi}, (a \vee b)^{\sim}_{\pi})$

$= (a_{\pi} b_{\pi} + a^{\sim}_{\pi} b_{\pi} + a_{\pi} b^{\sim}_{\pi}, a^{\sim}_{\pi} b^{\sim}_{\pi})$

$= (a_{\pi}, a^{\sim}_{\pi}) \vee (b_{\pi}, b^{\sim}_{\pi})$

$= f(a) \vee f(b)$

$\therefore f(a \vee b) = f(a) \vee f(b)$

$f(a^{\sim}) = (a^{\sim}_{\pi}, a^{\sim\sim}_{\pi})$

$= (a^{\sim}_{\pi}, a_{\pi})$

$= (a_{\pi}, a^{\sim}_{\pi})^{\sim}$

$= f(a^{\sim})$

$\therefore f(a^{\sim}) = f(a)^{\sim}$

$\therefore A \cong A(B(A))$

3.8 Theorem: Let A_1, A_2 be Pre-A*algebra, B_1, B_2 be Boolean algebra,
Then (i) $A_1 \cong A_2$ iff $B(A_1) \cong B(A_2)$
(ii) $B_1 \cong B_2$ iff $A(B_1) \cong A(B_2)$

Proof: First we prove the following.

(a) $A_1 \cong A_2 \Rightarrow B(A_1) \cong B(A_2)$

(b) $B_1 \cong B_2 \Rightarrow A(B_1) \cong A(B_2)$

(a) $A_1 \cong A_2$

Let: $f: A_1 \to A_2$ be a Pre-A*algebra isomorphism

Let $a \in B(A_1) \Rightarrow \exists x \in A_1 \ni a = x_{\pi}$

$f(a) = f(x_{\pi}) = f(x_{\pi}) \in B(A_2)$

Let $b \in B(A_2) \Rightarrow \exists\, y \in A_2 \ni b = y_{\pi}$

$\because f: A_1 \to A_2$ is isomorphism and $y \in A_2$

$\Rightarrow \exists\, x \in A_1 \ni f(x) = y$

$f(x_{\pi}) = (f(x))_{\pi} = y_{\pi} = b$

$\therefore a \in B(A_1) \Leftrightarrow f(a) \in B(A_2)$

$\therefore$ f: $B(A_1) \to B(A_2)$ is a Boolean isomorphism

$\therefore B(A_1) \cong B(A_2)$

(b) Suppose $B_1 \cong B_2$

Let: $f: B_1 \to B_2$ be a Boolean isomorphism

Let $a, b \in B_1$, $a \wedge b = 0 \Rightarrow f(a) \wedge f(b) = 0$

Define g: $A(B_1) \to A(B_2)$ as follows

Let $(a, b) \in A(B_1) \Rightarrow (a, b) \in B_1$, $a \wedge b = 0$

$g(a, b) = f(a), f(b) \in A(B_2)$

$\therefore g$ is well defined and g is bisection

$g[(a, b) \wedge (x, y)] = g(ax, ay + bx + by)$

$= [f(ax), f(ay+bx+by)$

$= [f(a) f(x), f(a) f(y) + f(b) f(x) + f(b) f(y)]$

$= (f(a), f(b)) \wedge (f(x), f(y))$

$= g(a, b) \wedge g(x, y)$

$\therefore g[(a, b) \wedge (x, y)] = g(a, b) \wedge g(x, y)$

$g[(a, b)^{\sim}] = g(b, a) = (f(b), f(a))$

$= (f(a), f(b))^{\sim}$

$= (g(a, b))^{\sim}$

$\therefore g[(a, b)^{\sim}] = (g(a, b))^{\sim}$

$g[(a, b) \vee (x, y)] = g(ax + ay + bx, by)$

$= [f(ax + ay + bx), f(by)]$

$= [f(a) f(x) + f(a) f(y) + f(b) f(x), f(b) f(y)]$

$= (f(a), f(b)) \vee (f(x), f(y))$

$= g(a, b) \vee g(x, y)$

$\therefore g[(a, b) \vee (x, y)] = g(a, b) \vee g(x, y)$

$\therefore A\ (B_1) \cong A\ (B_2)$

(i) From (a) $A_1 \cong A_2 \Rightarrow B\ (A_1) \cong B\ (A_2)$

Suppose $B\ (A_1) \cong B\ (A_2)$

$\Rightarrow A\ (B(A_1)) \cong A\ (B\ (A_2))$ by (b)

But $A\ (B\ (A_1)) \cong A_1$ by 2.2 by (ii)

$A\ (B\ (A_2)) \cong A_2$

$\therefore A_1 \cong A_2$

(ii) From (b) $B_1 \cong B_2 \Rightarrow A\ (B_1) \cong A(B_2)$

Suppose $A\ (B_1) \cong A\ (B_2)$

$\Rightarrow B\ (A\ (B_1)) \cong B\ (A\ (B_2))$

But $B\ (A\ (\ B_1\ )\ ) \cong B_1$

$B\ (A\ (B_2)) \cong B_2$

$\therefore B_1 \cong B_2$.

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